

ON IDEALS IN THE ENVELOPING ALGEBRA OF A LOCALLY SIMPLE LIE ALGEBRA

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ABSTRACT. We study (two-sided) ideals I in the enveloping algebra $U(\mathfrak{g}_\infty)$ of an infinite-dimensional Lie algebra \mathfrak{g}_∞ obtained as the union (equivalently, direct limit) of an arbitrary chain of embeddings of simple finite-dimensional Lie algebras

$$\mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \rightarrow \dots \rightarrow \mathfrak{g}_n \rightarrow \dots$$

with $\lim_{n \rightarrow \infty} \dim \mathfrak{g}_n = \infty$. Our main result is an explicit description of the zero-sets of the corresponding graded ideals $\text{gr } I$. We use this description and results of A. Zhilinskii to prove Baranov's conjecture that, if \mathfrak{g}_∞ is not diagonal in the sense of A. Baranov and A. Zhilinskii, then $U(\mathfrak{g}_\infty)$ admits a single non-zero proper ideal: the augmentation ideal.

Our study is based on a complete description of the radical Poisson ideals in $\mathbf{S}^*(\mathfrak{g}_\infty)$ and their zero-sets. We then discuss in detail integrable ideals of $U(\mathfrak{g}_\infty)$, i.e. ideals $I \subset U(\mathfrak{g}_\infty)$ for which $I \cap U(\mathfrak{g}_n)$ is an intersection of ideals of finite-codimension in $U(\mathfrak{g}_n)$ for any $n \geq 1$. We present a classification of prime integrable ideals based on work of A. Zhilinskii. For $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty$, all zero-sets of radical Poisson ideals of $\mathbf{S}^*(\mathfrak{g}_\infty)$ arise from prime integrable ideals of $U(\mathfrak{g}_\infty)$. For $\mathfrak{g}_\infty \cong \mathfrak{sp}_\infty$ only "half" of the zero-sets of Poisson ideals $\mathbf{S}^*(\mathfrak{g}_\infty)$ arise from integrable ideals of $U(\mathfrak{g}_\infty)$.

Key words: associated variety, coherent local system, integrable ideal, locally finite Lie algebra, moment map, Poisson ideal.

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1. INTRODUCTION

We work over an algebraically closed field of characteristic zero. A *locally finite Lie algebra* is by definition a Lie algebra isomorphic to the limit of some system of finite-dimensional Lie algebras [BS]. In what follows we restrict ourselves to locally finite Lie algebras \mathfrak{g}_∞ defined as direct limits of respective sequences of embeddings of simple Lie algebras

$$\mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \rightarrow \dots \rightarrow \mathfrak{g}_n \rightarrow \dots \tag{1}$$

such that $\lim_{n \rightarrow \infty} \dim \mathfrak{g}_n = \infty$; for brevity we refer just to these Lie algebras as *locally simple Lie algebras*.

Locally simple Lie algebras are natural infinite-dimensional analogs of finite-dimensional simple Lie algebras, and their representation theory is interesting and challenging.

In this paper we study ideals in the universal enveloping algebra $U(\mathfrak{g}_\infty)$ of a locally simple Lie algebra \mathfrak{g}_∞ , in particular annihilators of simple \mathfrak{g}_∞ -modules, i.e. primitive ideals. The structure of ideals in $U(\mathfrak{g}_\infty)$ differs significantly from the structure of ideals in the enveloping algebra of a simple finite-dimensional Lie

algebra. This is not surprising once one observes that the center of $U(\mathfrak{g}_\infty)$ consists of constants only.

The theory of ideals in $U(\mathfrak{g}_\infty)$ has been initiated by A. Zhilinskii in [Zh1], [Zh2]. His method is to study “coherent local systems” of finite-dimensional modules of the sequence of Lie algebras \mathfrak{g}_n and to construct ideals in $U(\mathfrak{g}_\infty)$ whose intersections with $U(\mathfrak{g}_n)$ are the joint annihilators of the modules at the n -th level of the respective local systems. We call the ideals of $U(\mathfrak{g}_\infty)$ arising in this way *integrable* ideals.

A central idea of the present paper is to gain information about ideals in $U(\mathfrak{g}_\infty)$ by studying their associated “varieties”. At first this leads to the study of radical Poisson ideals in the symmetric algebra $\mathbf{S}(\mathfrak{g}_\infty)$. Our first notable result is that $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal J of locally infinite codimension (i.e. such that $J \cap \mathbf{S}(\mathfrak{g}_n)$ is of infinite codimension in $\mathbf{S}(\mathfrak{g}_n)$ for almost all n) if and only if \mathfrak{g}_∞ is isomorphic to one of the three locally simple Lie algebras $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

This result, together with results of A. Zhilinskii [Zh2], yields the following corollary: if \mathfrak{g} is not isomorphic to $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, and $U(\mathfrak{g}_\infty)$ has a non-zero proper ideal I which does not coincide with the augmentation ideal, then I is of locally finite codimension (i.e. $I \cap U(\mathfrak{g}_n)$ is of finite codimension in $U(\mathfrak{g}_n)$ for all n) and \mathfrak{g}_∞ is diagonal. Diagonal locally simple Lie algebras form a natural class of locally simple Lie algebras which has been introduced by A. Baranov and A. Zhilinskii; in fact, these authors have given an intricate and elegant classification of diagonal locally simple Lie algebras [BZh]. Moreover, the above corollary had been conjectured by A. Baranov.

As a second notable result on Poisson ideals we compute the set of zeros $\text{Var}(J)$ (i.e. the associated “variety”) of any Poisson ideal $J \subset \mathbf{S}(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

We then pass to the study of ideals in $U(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. In Section 7 we review Zhilinskii’s results and, most importantly, his classification of irreducible coherent local systems. Zhilinskii’s results yield an explicit description of all prime integrable ideals in $U(\mathfrak{g}_\infty)$, Theorem 7.8.

We finally describe the associated “varieties” of integrable ideals. In particular, we show that if $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and J is a radical Poisson ideal of $\mathbf{S}(\mathfrak{g}_\infty)$, there exists a prime integrable ideal I of $U(\mathfrak{g}_\infty)$ such that $\text{Var}(I) = \text{Var}(J)$. For $\mathfrak{g}_\infty = \mathfrak{sp}_\infty$ the situation is different: given an ideal J , there exists an ideal I of $U(\mathfrak{g}_\infty)$ such that $\text{Var}(I) = \text{Var}(J)$, but I is not necessarily integrable.

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2. PRELIMINARIES

We fix an algebraically closed field \mathbb{F} of characteristic zero. All vector spaces (including Lie algebras) are assumed to be defined over \mathbb{F} . If V is a vector space, V^* stands for the dual space $\text{Hom}_{\mathbb{F}}(V, \mathbb{F})$. All varieties we consider are algebraic varieties over \mathbb{F} (with Zariski topology). When considering locally finite Lie algebras or their enveloping algebras we assume that any given sequence (1) consists of inclusions, so we can freely interchange $\varinjlim \mathfrak{g}_n$ with $\cup_n \mathfrak{g}_n$ and $\varinjlim U(\mathfrak{g}_n)$ with $\cup_n U(\mathfrak{g}_n)$.

There is no classification of general locally simple Lie algebras: a classification is only available for the so called diagonal locally simple Lie algebras, see [BZh].

Among diagonal locally simple Lie algebras a prominent role is played by the three simple Lie algebras \mathfrak{sl}_∞ , \mathfrak{so}_∞ and \mathfrak{sp}_∞ which can be defined as unions of the respective chains of inclusions of classical finite-dimensional Lie algebras of types \mathfrak{sl} , \mathfrak{so} , or \mathfrak{sp} under the obvious “left upper-corner inclusions”. An important result, see [B] or [BS], states that, up to isomorphism, these three Lie algebras are the only locally simple finitary Lie algebras, i.e. locally simple Lie algebras which admit a countable-dimensional faithful module with a basis such that the endomorphism arising from each element of the Lie algebra is given by a matrix with finitely many non-zero entries.

Let G be a connected algebraic group with Lie algebra, \mathfrak{g} and $I \subset U(\mathfrak{g})$ be an ideal in the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} (by an ideal in a ring we mean a proper two-sided ideal). The degree filtration $\{U(\mathfrak{g})^{\leq d}\}_{d \in \mathbb{Z}_{\geq 0}}$ on $U(\mathfrak{g})$ defines the filtration $\{I \cap U(\mathfrak{g})^{\leq d}\}_{d \in \mathbb{Z}_{\geq 0}}$ on I . The associated graded object $\text{gr} I := \bigoplus_d (I \cap U(\mathfrak{g})^{\leq d}) / (I \cap U(\mathfrak{g})^{\leq d-1})$ is a G -stable ideal of $\text{gr} U(\mathfrak{g}) = \mathbf{S}^*(\mathfrak{g})$. We denote the set of zeros of $\text{gr} I$ in \mathfrak{g}^* by $\text{Var}(I) \subset \mathfrak{g}^*$. The variety $\text{Var}(I)$ is a G -stable subvariety of \mathfrak{g}^* and is, by definition, the *associated variety* of I . The ideal I is of finite codimension in $U(\mathfrak{g})$ if and only if $\text{gr} I$ is of finite codimension in $\mathbf{S}^*(\mathfrak{g})$. Moreover, $\text{gr} I$ is of finite codimension in $\mathbf{S}^*(\mathfrak{g})$ if and only if $\text{Var}(I) = 0$. We note that the radical $\text{rad}(\text{gr} I)$ of $\text{gr} I$ is also G -stable.

It is well known that $\mathbf{S}^*(\mathfrak{g})$ is a Poisson algebra, i.e. $\mathbf{S}^*(\mathfrak{g})$ has an \mathbb{F} -bilinear operation

$$\{\cdot, \cdot\}: \mathbf{S}^*(\mathfrak{g}) \times \mathbf{S}^*(\mathfrak{g}) \rightarrow \mathbf{S}^*(\mathfrak{g})$$

which is a Lie bracket and is compatible with multiplication. An ideal $J \subset \mathbf{S}^*(\mathfrak{g})$ is *Poisson* whenever $\{f, J\} \subset J$ for any $f \in \mathbf{S}^*(\mathfrak{g})$. Assume that \mathfrak{g} is finite-dimensional and semisimple and let G be the adjoint group of \mathfrak{g} . Then the radical Poisson ideals of $\mathbf{S}^*(\mathfrak{g})$ are in one-to-one correspondence with the G -stable Zariski-closed subsets of \mathfrak{g}^* . A description of such sets is presented in [Bor].

Let \mathfrak{g}_∞ be a locally simple Lie algebra. The same arguments as in the previous paragraph assign to any ideal $I \subset U(\mathfrak{g}_\infty)$ a radical ideal $\text{rad}(\text{gr} I) \subset \mathbf{S}^*(\mathfrak{g}_\infty)$, which is stable under the adjoint action of \mathfrak{g}_∞ on $\mathbf{S}^*(\mathfrak{g}_\infty)$; in what follows we call \mathfrak{g}_∞ -stable ideals of $\mathbf{S}^*(\mathfrak{g}_\infty)$ Poisson. We denote the set of zeros of $\text{gr} I$ in \mathfrak{g}_∞^* by $\text{Var}(I)$ and refer to $\text{Var}(I)$ as the associated “variety” of I .

Note that $\text{Var}(I)$ is a proj-variety, i.e. an inverse limit of algebraic varieties. Indeed, fix a sequence (1) with $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$ and let $\overline{\text{pr}_{\mathfrak{g}_n} \text{Var}(I)} \subset \mathfrak{g}_n^*$ be the closure of the image of $\text{Var}(I)$ under the natural projection $\text{pr}_{\mathfrak{g}_n}: \mathfrak{g}_\infty^* \rightarrow \mathfrak{g}_n^*$; by definition $\overline{\text{pr}_{\mathfrak{g}_n} \text{Var}(I)} \subset \mathfrak{g}_n^*$ is the set of zeros of $(\text{gr} I) \cap \mathbf{S}^*(\mathfrak{g}_n)$ in \mathfrak{g}_n^* . The space \mathfrak{g}_∞^* equals the inverse limit $\varprojlim \mathfrak{g}_n^*$, and therefore $\text{Var}(I) \subset \mathfrak{g}_\infty^*$ is the inverse limit of the algebraic varieties $\overline{\text{pr}_{\mathfrak{g}_n} \text{Var}(I)}$.

An ideal I is of *locally finite codimension* in $U(\mathfrak{g}_\infty)$ (i.e. $I \cap U(\mathfrak{g}_n)$ is of finite codimension in $U(\mathfrak{g}_n)$ for all n) if and only if the ideal $\text{gr} I$ is of *locally finite codimension* in $\mathbf{S}^*(\mathfrak{g}_\infty)$ (i.e. $(\text{gr} I) \cap \mathbf{S}^*(\mathfrak{g}_n)$ is of finite codimension in $\mathbf{S}^*(\mathfrak{g}_n)$ for all n). Furthermore, $\text{gr} I$ is of locally finite codimension in $\mathbf{S}^*(\mathfrak{g}_\infty)$ if and only if $\text{Var}(I) = 0$. We call an ideal I of *locally infinite codimension* if I is not of locally finite codimension.

The above discussion implies in particular the following.

Proposition 2.1. Assume that \mathfrak{g}_∞ is a locally simple Lie algebra. If $U(\mathfrak{g}_\infty)$ admits a non-zero ideal of locally infinite codimension, then $\mathbf{S}^*(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal of locally infinite codimension.

3. POISSON IDEALS: STATEMENT OF RESULTS

Our first main result is the following theorem.

Theorem 3.1. If $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal of locally infinite codimension, then $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

Corollary 3.2. If $\mathbf{U}(\mathfrak{g}_\infty)$ admits a non-zero ideal of locally infinite codimension, then $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

Proof. The algebra $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal J of locally infinite codimension by Proposition 2.1, hence $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ by Theorem 3.1. \square

Fix now a Lie algebra $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ together with a chain (1) such that $\varinjlim \mathfrak{g}_n = \mathfrak{g}_\infty$. Without loss of generality we assume that for $n \geq 3$ all \mathfrak{g}_n are simple and of the same type A, B, C, or D, and that $\text{rk } \mathfrak{g}_n = n$. By V_n we denote a natural representation of \mathfrak{g}_n (for \mathfrak{g}_n of type A there are two choices of V_n up to isomorphism). We further assume that, for $n \geq 3$, V_{n+1} considered as a \mathfrak{g}_n -module is isomorphic to V_n plus a trivial module.

Set

$$\mathfrak{g}_n^{\leq r} := \{x \in \mathfrak{g}_n \mid \text{there exists } \lambda \in \mathbb{F} \text{ such that } \text{rk}(X - \lambda \text{Id}_{V_n}) \leq r\}, \quad (2)$$

where X is considered as a linear operator on V_n . Note that $\mathfrak{g}_n^{\leq r}$ is a Zariski closed subset of \mathfrak{g}_n . Choosing compatible identifications $\mathfrak{g}_n \cong \mathfrak{g}_n^*$, we can assume that $\mathfrak{g}_n^{\leq r} \subset \mathfrak{g}_n^*$. Furthermore, for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ one can check directly that the projection $\mathfrak{g}_{n+1}^* \rightarrow \mathfrak{g}_n^*$ maps $\mathfrak{g}_{n+1}^{\leq r}$ surjectively onto $\mathfrak{g}_n^{\leq r}$. This yields a well-defined limit of algebraic varieties $\varprojlim \mathfrak{g}_n^{\leq r}$ which we denote by $\mathfrak{s}_\infty^{\leq r}$, where \mathfrak{s} is an abbreviation for $\mathfrak{sl}, \mathfrak{so}$ or \mathfrak{sp} .

The radical ideals $J_n^{\leq r}$ of $\mathbf{S}(\mathfrak{g}_n)$ with respective zero-sets $\mathfrak{g}_n^{\leq r} \subset \mathfrak{g}_n^*$ form a chain whose union we denote by $J^{\leq r}$. The ideal $J^{\leq r}$ is a radical Poisson ideal of $\mathbf{S}(\mathfrak{g}_\infty)$. It turns out that any non-zero radical Poisson ideal of $\mathbf{S}(\mathfrak{g}_\infty)$ is of this form.

Theorem 3.3. Let $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ and $J \subset \mathbf{S}(\mathfrak{g}_\infty)$ be a non-zero radical Poisson ideal. Then $J = J^{\leq r}$ for some $r \in \mathbb{Z}_{\geq 0}$.

Theorems 3.1 and 3.3 are proved in Section 5.

4. AUXILIARY RESULTS

Throughout the paper V denotes a finite-dimensional vector space of dimension d_V . In order to consider all simple classical groups simultaneously, we use \mathbf{S} (respectively, \mathfrak{s}) as an abbreviation for $\text{SL}, \text{SO}, \text{Sp}$ (respectively, $\mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}$) and consider three different cases. In the case $\mathbf{S}=\text{SL}$ we fix the zero bilinear form on the space V and set $\mathbf{S}(V) := \text{SL}(V)$, $\mathfrak{s}(V) := \mathfrak{sl}(V)$. In the case $\mathbf{S}=\text{SO}$ we fix a nondegenerate symmetric bilinear form on V and set $\mathbf{S}(V) := \text{SO}(V)$, $\mathfrak{s}(V) := \mathfrak{so}(V)$. In the case $\mathbf{S}=\text{Sp}$ we assume that d_V is even and fix a nondegenerate antisymmetric bilinear form on V . Then $\mathbf{S}(V) := \text{Sp}(V)$, $\mathfrak{s}(V) := \mathfrak{sp}(V)$.

In this section G denotes a connected simple subgroup of $\mathbf{S}(V)$ with Lie algebra $\mathfrak{g} \subset \mathfrak{s}(V)$. We start with some general statements about G -orbits in \mathfrak{g}^* (i.e. about coadjoint orbits). We identify \mathfrak{g} and \mathfrak{g}^* via the Cartan-Killing form. Fix $e \in \mathfrak{g}^*$. Denote the G -orbit of e in \mathfrak{g}^* by $\mathcal{O}[e]$. Let $\underline{\mathcal{O}}[e]$ be the unique closed G -orbit of the closure $\overline{\mathcal{O}}[e]$ in \mathfrak{g}^* . By the Luna Slice Theorem [VP], there exists a G -equivariant morphism $\mathcal{O}[e] \rightarrow \underline{\mathcal{O}}[e]$. Assume that $\underline{\mathcal{O}}[e] \neq 0$, i.e. e is not nilpotent. Let $h \in \underline{\mathcal{O}}[e]$

be a semisimple element. Then the centralizer of h in \mathfrak{g} is a Levi subalgebra¹ of some parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Let $P \subset G$ be a parabolic subgroup with Lie algebra \mathfrak{p} . Then there exists a finite G -equivariant covering $\tilde{\mathcal{O}} \rightarrow \mathcal{O}[e]$ and a G -equivariant morphism $\tilde{\mathcal{O}} \rightarrow G/P$.

Assume now that $\mathcal{Q}[e] = \{0\}$, i.e. that e is nilpotent. By the Jacobson-Morozov Theorem there exist elements h, f such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h,$$

i.e. such that $\{e, h, f\}$ is an \mathfrak{sl}_2 -triple. The element h is rational semisimple. Hence \mathfrak{g} splits into the direct sum of $\text{ad}h$ -eigenspaces $\oplus_{i \in \mathbb{Q}} \mathfrak{g}_i$ with rational eigenvalues. The direct sum $\mathfrak{g}_h^+ := \oplus_{i \geq 0} \mathfrak{g}_i \subset \mathfrak{g}$ is a parabolic subalgebra of \mathfrak{g} and we denote it by \mathfrak{p}_e . The subalgebra \mathfrak{p}_e is determined by e . By $P_e \subset G$ we denote the parabolic subgroup with the Lie algebra \mathfrak{p}_e . There is a G -equivariant morphism $\mathcal{O}[e] \rightarrow G/P_e$.

The above discussion is summarized in the following lemma.

Lemma 4.1. A suitable finite covering of any coadjoint orbit admits a G -equivariant (and thus surjective) morphism to G/P , where P is a maximal parabolic subgroup.

For $\mathfrak{g} = \mathfrak{sl}(V)$, a quotient G/P is nothing but a Grassmannian $\text{Gr}(r; V)$ for some $r < d_V$. For $\mathfrak{g} = \mathfrak{so}(V)$ and $\mathfrak{sp}(V)$ a quotient G/P is an irreducible component of the variety $\text{Gr}^{(0)r}(V)$ of isotropic subspaces in V of dimension r for some $r \leq \frac{d_V}{2}$. The variety $\text{Gr}^{(0)r}(V)$ is irreducible unless $\mathfrak{g} = \mathfrak{so}(V)$ and $r = \frac{d_V}{2}$. In this latter case $\text{Gr}^{(0)r}(V)$ has two irreducible components which are isomorphic as varieties. More generally, for $k < r < d_V$, we denote by $\text{Gr}^{(k)r}(V)$ the variety of subspaces of V of dimension r on which the restriction of the fixed form on V has rank k .

For $X \in \text{End}V$ we denote by $V_\lambda^X \subset V$ the generalized eigenspace of X with eigenvalue $\lambda \in \mathbb{F}$. Furthermore, $\mathfrak{g} \cdot V$ stands for the sum of the non-trivial simple \mathfrak{g} -submodules of V .

Proposition 4.2. Assume that there is an $S(V)$ -orbit \mathcal{O} in $\mathfrak{s}(V)^*$ such that its image in \mathfrak{g}^* under the canonical projection $\mathfrak{s}(V)^* \rightarrow \mathfrak{g}^*$ is not dense. Then

$$\dim(\mathfrak{g} \cdot V) < 2(\dim G - \text{rk } G)(\text{rk } G + 1) \text{ or } 2\dim G + 2 \geq d_V.$$

In order to prove Proposition 4.2 we need several preliminary statements.

Lemma 4.3. Let V' be any subspace in V of dimension $d_{V'}$.

- a) Assume that $\mathfrak{s} = \mathfrak{sl}$, \mathfrak{sp} . If $\dim(V^r \cap V') \geq 1$ for any isotropic subspace $V^r \subset V$ of dimension r , then $r + d_{V'} > d_V$.
- b) Assume that $\mathfrak{s} = \mathfrak{so}$ and $r < \frac{d_V}{2}$. If $\dim(V^r \cap V') \geq 1$ for any isotropic subspace $V^r \subset V$ of dimension r , then $r + d_{V'} > d_V$.
- c) Assume that $\mathfrak{s} = \mathfrak{so}$, d_V is even, and $r = \frac{d_V}{2}$. If $\dim(V^r \cap V') \geq 1$ for any isotropic subspace $V^r \subset V$ of dimension r from some irreducible component of $\text{Gr}^{(0)r}(V)$, then either $r + d_{V'} > d_V$, or $r + d_{V'} = d_V$ and V' is isotropic.

Proof. Exercise in linear algebra. □

Let Z be a G -variety. We denote the maximal dimension of a G -orbit on Z by $m_G(Z)$. For a subvariety $Y \subset Z$ we set

$$GY := \{z \in Z \mid z = gy \text{ for some } y \in Y \text{ and } g \in G\}.$$

¹Under a Levi subalgebra of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ we understand a maximal reductive in \mathfrak{g} subalgebra of \mathfrak{p} .

Lemma 4.4. a) Assume that $m_G(\mathbb{P}(V)) < \dim G$. Then

$$\dim(\mathfrak{g} \cdot V) < 2(\dim G - \text{rank} G)(\text{rank} G + 1).$$

b) Let $\mathfrak{s} = \mathfrak{so}$. Assume that

$$m_G(\mathbb{P}(V)) = \dim G \text{ and } m_G(\text{Gr}^0(1; V)) < \dim G.$$

Then

$$2\dim G + 2 \geq d_V.$$

Proof. The statement of part a) is the main result of [AnPo]. In part b) $\text{Gr}^0(1; V) = \mathbb{P}(\mathcal{V})$, where \mathcal{V} is the set of isotropic vectors in V . The inequality $m_G(\mathbb{P}(\mathcal{V})) < \dim G$ implies that the stabilizer of a generic point of $\mathbb{P}(\mathcal{V})$ under the action of G has positive dimension. Therefore there exists $A \in \mathfrak{g}$ such that $\mathcal{O}[A]$ intersects \mathfrak{g}_x for all x in an open subset of $\mathbb{P}(\mathcal{V})$. Then, for some eigenvalue λ of A , we have $\mathcal{V} \subset \overline{GV_\lambda^A}$, where

$$V_\lambda^A := \text{Ker}(A - \lambda \text{Id}_V)^{d_V}$$

and $\overline{GV_\lambda^A}$ is the closure of GV_λ^A in V . Furthermore, the equality $m_G(\mathbb{P}(V)) = \dim G$ implies that \mathcal{V} coincides with $\overline{GV_\lambda^A}$. Therefore $V_\lambda^A \subset \mathcal{V}$, and the space V_λ^A is isotropic. In particular, $\dim V_\lambda^A \leq \frac{d_V}{2}$, and consequently $\dim G \geq \frac{d_V}{2} - 1$. \square

Lemma 4.5. Let

$$2 \leq r \leq \frac{d_V}{2}, m_G(\mathbb{P}(V)) = \dim G \text{ and } m_G(\text{Gr}^0(r; V)) < \dim G$$

for some r . Then

$$2\dim G + 2 \geq d_V.$$

Proof. Let $x \in \text{Gr}^0(r; V)$ denote a point and $V(x)$ be the corresponding r -dimensional space. Let G_x be the stabilizer of x in G ; G_x acts on $V(x)$ and, if x is generic, we have

$$m_{G_x}(\mathbb{P}(V(x))) = \dim G_x.$$

Let $p \in V(x)$ be a non-zero vector, $\langle p \rangle \subset V(x)$ be the line generated by p , and $\mathfrak{g}p \subset V$ be the tangent space to Gp in p . Then $\dim((V(x)/\langle p \rangle) \cap \mathfrak{g}\langle p \rangle) \geq \dim G_x$, where $\mathfrak{g}\langle p \rangle$ is the image of $\mathfrak{g}p$ in $V/\langle p \rangle$. Hence, for a generic $\langle p \rangle \in \text{Gr}^0(1; V)$ and any

$$\tilde{V} \in \text{Gr}^0(r-1; \langle p \rangle^\perp / \langle p \rangle)$$

we have $\dim(\tilde{V} \cap \mathfrak{g}\langle p \rangle) \geq \dim G_x$. In particular, $\dim(\tilde{V} \cap \mathfrak{g}\langle p \rangle) \geq 1$. Therefore, by Lemma 4.3 (applied to $V' = \mathfrak{g}\langle p \rangle$ and $V = \langle p \rangle^\perp / \langle p \rangle$) we obtain

$$r - 1 + \dim \mathfrak{g} \geq \dim(\langle p \rangle^\perp / \langle p \rangle).$$

As $\dim(\langle p \rangle^\perp / \langle p \rangle) \geq d_V - 2$ and $r \leq \frac{d_V}{2}$, we conclude that $2\dim G + 2 \geq d_V$. \square

Proposition 4.6 ([Vi, text after Prop. 3]). Let Z be an irreducible G -symplectic variety with a moment map $\Phi : Z \rightarrow \mathfrak{g}^*$. Then the dimension of $\Phi(Z)$ equals $m_G(Z)$.

Proof of Proposition 4.2. Since we can consider the morphism $\mathcal{O} \rightarrow \mathfrak{g}^*$ as a moment map, Proposition 4.6 implies that the image of \mathcal{O} in \mathfrak{g}^* is not dense if and only if

$$m_G(\mathcal{O}) < \dim G.$$

Furthermore, according to Lemma 4.1, for some r , $1 \leq r \leq \frac{d_V}{2}$, there exists an $S(V)$ -equivariant morphism from a finite covering of \mathcal{O} onto $\text{Gr}^0(r; V)$. Therefore

$$m_G(\text{Gr}^{(0)r}(V)) < \dim G.$$

If $m_G(\text{Gr}^{(0)1}(V)) < \dim G$, the claim of Proposition 4.2 follows from Lemma 4.4. Indeed, if also $m_G(\mathbb{P}(V)) < \dim G$, we apply Lemma 4.4 a), and if $m_G(\mathbb{P}(V)) = \dim G$, then $\mathfrak{s} = \mathfrak{so}$ (as $\mathbb{P}(V) \neq \text{Gr}^{(0)1}(V)$) and we apply Lemma 4.4 b).

If

$$m_G(\text{Gr}^{(0)1}(V)) = \dim G,$$

then necessarily $m_G(\mathbb{P}(V)) = \dim G$ and $r \geq 2$. In this case the claim of Proposition 4.2 follows from Lemma 4.5. \square

Proposition 4.2 is used in a crucial way in the proof of Theorem 3.1. The results in the remaining part of this section are necessary for the proof of Theorem 3.3.

In what follows W denotes a subspace of V of dimension d_W . If $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$, we assume that the restriction of the fixed form on V is nondegenerate on W . This yields embeddings $S(W) \rightarrow S(V)$ and $\mathfrak{s}(W) \rightarrow \mathfrak{s}(V)$. For $\mathfrak{s} = \mathfrak{sl}$ an embedding $S(W) \rightarrow S(V)$ is determined by a choice of complement to W in V . Moreover, in all three cases we have an orthogonal (with respect to the Cartan-Killing form on $\mathfrak{s}(V)$) projection $\phi : \mathfrak{s}(V) \rightarrow \mathfrak{s}(W)$. It is easy to see that $\phi(X) = \text{pr}_W \circ (X|_W)$, where $X \in \mathfrak{s}(V)$ is viewed as an element of $\text{End } V$ and $\text{pr}_W : V \rightarrow W$ is the orthogonal projection for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$, and respectively the projection along the fixed complement of W in V for $\mathfrak{s} = \mathfrak{sl}$. We fix the embedding $\mathfrak{s}(W) \rightarrow \mathfrak{s}(V)$, and $\phi : \mathfrak{s}(V) \rightarrow \mathfrak{s}(W)$ stands for the corresponding projection.

Recall that $\mathcal{O}[X]$ denotes the $S(V)$ -orbit in $\mathfrak{s}(V)$ of $X \in \mathfrak{s}(V)$.

Lemma 4.7. Let $X \in \mathfrak{s}(V)$. Then

- a) $\text{rk } \phi(X) \leq \text{rk } X$;
- b) if $d_W > \text{rk } X$, the image $\phi(\mathcal{O}[X])$ contains an element of rank $\text{rk } X$.

Proof. Part a) follows immediately from the formula

$$\phi(X) = \text{pr}_W \circ (X|_W).$$

To prove part b) we consider a generic subspace $\tilde{W} \subset W$ of dimension d_W . Note that for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$ the restriction of the form to \tilde{W} is nondegenerate. Furthermore, $\text{rk } X|_{\tilde{W}} = \text{rk } X$, and $\tilde{W}^\perp \cap \text{Im } X = 0$ where $\text{Im } X$ is the image of $X \in \text{End } V$. Therefore $\text{rk}(\text{pr}_{\tilde{W}} \circ X|_{\tilde{W}}) = \text{rk } X$.

Since there exists $g \in S(V)$ with $g(W) = \tilde{W}$, we have

$$\text{rk } \phi(g(X)) = \text{rk } X.$$

\square

Proof. The proof is very similar to the proof of Lemma 4.7. \square

Lemma 4.8. Assume $r \leq \frac{d_W}{2}$. Then the set $\mathfrak{s}(V)^{\leq r}$ defined by (2) is the largest $S(V)$ -invariant subset of the preimage $\phi^{-1}(\mathfrak{s}(W)^{\leq r})$.

Proof. Exercise in linear algebra. \square

Lemma 4.9. Consider the projection $\varphi : \mathfrak{sp}(V)^* \rightarrow \mathfrak{gl}(V^{iso})^*$ dual to the embedding $\mathfrak{gl}(V^{iso}) \rightarrow \mathfrak{sp}(V)$, where V^{iso} is a maximal isotropic subspace of V . Assume $r \leq \frac{d_V}{2}$. Then the set $\mathfrak{sp}(V)^{\leq r}$ defined by (2) is the largest $\text{Sp}(V)$ -invariant subset of the preimage $\varphi^{-1}(\mathfrak{sl}(V^{iso})^{\leq r})$.

Proof. Exercise in linear algebra. \square

Lemma 4.10. Let $x \in \text{Gr}(r; V)$ and let $V(x) \subset V$ be the corresponding subspace. Let W' be the $S(W)$ -stable complement to W in V . Assume that $d_V \geq 2d_W$, $d_W \leq r \leq d_V - d_W$, and $\dim(V(x) \cap W') = r - d_W$. Then the stabilizer of $x \in \text{Gr}(r; V)$ in $\text{SL}(W)$ (and therefore also in $\text{SO}(W)$, $\text{Sp}(W)$) is trivial.

Proof. Exercise in linear algebra. \square

For a subset $S \subset \mathbb{F}$ we set $V_S^X := \oplus_{\lambda \in S} V_\lambda^X$. We set also $\hat{0} := \mathbb{F} \setminus \{0\}$; then $V = V_0^X \oplus V_{\hat{0}}^X$. Let $X_{nn} := X|_{V_0}$, $X_r := X|_{V_{\hat{0}}}$, and $X_r = X_s + X_{ns}$ be the Jordan decomposition as sum of commuting semisimple and nilpotent elements. The decomposition $V = V_0^X \oplus V_{\hat{0}}^X$ allows us to consider all four operators X_{nn}, X_r, X_s, X_{ns} as endomorphisms of V . Furthermore,

$$\text{rk}(X_s + X_{ns}) = \text{rk } X_s, \quad \text{rk}(X_s + X_{nn}) = \text{rk } X,$$

and $X_s + X_{nn} \in \overline{\mathcal{O}[X]}$.

Definition 4.11. We say that $X \in \text{End } V$ is *rank-reduced* whenever

$$X_{nn}^2 = 0 \text{ and } X_{ns} = 0.$$

Lemma 4.12. If for some $X \in \mathfrak{s}(V)$ we have $2\text{rk } X \leq d_V$, then there exists a (unique up to conjugation) rank-reduced element $X' \in \mathfrak{s}(V)$ such that $X' \in \overline{\mathcal{O}[X]}$ and $\text{rk } X = \text{rk } X'$.

Proof. The condition $(X'_{nn})^2 = 0$ means that X'_{nn} is nilpotent and the sizes of its Jordan blocks are at most 2×2 . Moreover, the number of non-zero Jordan blocks of X'_{nn} equals $\text{rk } X'_{nn}$. In particular, all nilpotent rank-reduced endomorphisms in $\mathfrak{s}(V)$ of fixed rank are conjugate (possibly, by outer automorphisms of $\mathfrak{s}(V)$). Since $2\text{rk } X_{nn} \leq d_V$, there exists a nilpotent rank-reduced endomorphism X_{rr} of rank equal to $\text{rk } X$ and such that $X_{rr} \in \overline{\mathcal{O}[X_{nn}]}$ [CM]. A rank-reduced endomorphism $X' \in \mathfrak{s}(V)$ such that $X'_s = X_s$ and X'_{nn} is conjugate to X_{rr} , is as desired. By construction, X' is unique up to conjugation. \square

Lemma 4.13. Assume $d_V > 3d_W$. If for some $X \in \mathfrak{s}(V)$ the image $\phi(\mathcal{O}[X])$ is not dense in $\mathfrak{s}(W)$, then there exists a unique $\lambda \in \mathbb{F}$ such that

$$\text{rk}(X - \lambda \text{Id}_V) < d_W.$$

Proof. Set $d_V(\mu) := \dim V_\mu^X$ and $d_V(S) := \sum_{\mu \in S} d_V(\mu)$ for any subset $S \subset \mathbb{F}$. We have an $S(V)$ -equivariant morphism

$$\mathcal{O}[X] \rightarrow \text{Gr}(d_V(S); V), \quad X \mapsto \oplus_{\mu \in S} V_\mu^X.$$

As $\phi(\mathcal{O}[X])$ is not dense in $\mathfrak{s}(W)$, the inequality $\text{m}_{S(W)}(\mathcal{O}[X]) < \dim S(W)$ holds. Therefore, for any $S \subset \mathbb{F}$, there exists k such that

$$\text{m}_{S(W)}(\text{Gr}^k(d_V(S); V)) < \dim S(W). \quad (3)$$

We now show that, for any $S \subset \mathbb{F}$, either $d_V(S) < d_W$ or $d_V(S) > d_V - d_W$. Assume to the contrary that $d_W \leq d_V(S) \leq d_V - d_W$. Let $x \in \text{Gr}^k(d_V(S); V)$ be a generic point and $V(x) \subset V$ be the corresponding subspace. Then

$$\dim(V(x) \cap W') = \dim V(x) - d_W,$$

where W' is the $S(W)$ -stable complement to W in V , and Lemma 4.10 implies that the stabilizer of x in $S(W)$ is trivial. This contradicts (3).

We claim next that there exists $\lambda \in \mathbb{F}$ such that $d_V(\lambda) > d_V - d_W$. Indeed, assuming that $d_V(\lambda) < d_W$ for all $\lambda \in \mathbb{F}$, we see that the inequality $3d_W < d_V$, together with the alternative $d_V(S) < d_W$ or $d_V(S) > d_V - d_W$, implies $d_V(S) < d_W$ for any finite set S . This is again a contradiction.

Fix now λ with $d_V(\lambda) > d_V - d_W$. It is easy to see that $\text{rk}(X - \lambda \text{Id}_V) = d_V(\mathbb{C} \setminus \lambda) + \text{rk}(X - \lambda \text{Id}_V)|_{V_\lambda^X}$. Furthermore, we have a decomposition

$$X - \lambda \text{Id}_V = (X - \lambda \text{Id}_V)_s + (X - \lambda \text{Id}_V)_{ns} + (X - \lambda \text{Id}_V)_{nn}.$$

Let $r' \in \mathbb{Z}_{\geq 0}$ for $\mathfrak{s} = \mathfrak{sl}$ and $r' \in 2\mathbb{Z}_{\geq 0}$ for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$. If $r' < \text{rk}((X - \lambda \text{Id}_V)_{nn}|_{V_\lambda^X})$, there exists $X_{r'}^\lambda \in \overline{\mathcal{O}[(X - \lambda \text{Id}_V)_{nn}|_{V_\lambda^X}]} \subset \mathfrak{s}(V_\lambda^X)$ with $\text{rk } X_{r'}^\lambda = r'$. Hence, if $\text{rk}((X - \lambda \text{Id}_V)_s) \leq r' \leq \text{rk}(X - \lambda \text{Id}_V)$, there exists

$$X_{r'} \in \overline{\mathcal{O}[X]} \subset \mathfrak{s}(V)$$

with $X_{r'} - \lambda \text{Id} = (X - \lambda \text{Id})_s + X_{r' - \text{rk}((X - \lambda \text{Id})_s)}^\lambda$, and thus $\text{rk}(X_{r'} - \lambda \text{Id}_V) = r'$.

As, for any d with $d_W \leq d \leq d_V - d_W$, the equality $\text{ms}_{S(W)}(\text{Gr}^k d; V) = \dim S(W)$ holds for some k , for any r' we have either $\dim \text{Im}(X_{r'} - \lambda \text{Id}_V) < d_W$ or $\dim \text{Im}(X_{r'} - \lambda \text{Id}_V) > d_V - d_W$. Therefore, either $\text{rk}(X - \lambda \text{Id}_V) < d_W$ or $\text{rk}(X - \lambda \text{Id}_V)_s > d_V - d_W > d_W$. On the other hand, $\text{rk}(X - \lambda \text{Id}_V)_s < d_W$. This implies

$$\text{rk}(X - \lambda \text{Id}_V) < d_W.$$

□

Lemma 4.14. Assume $d_V \geq 2d_W$, $d_W > r$, and let $X \in \mathfrak{s}(V)^{\leq r}$. If $\text{rk}(X - \lambda \text{Id}_V) = r$ for some $\lambda \in \mathbb{F}$, then $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{s}(W)^{\leq r}$.

We consider separately the cases $\mathfrak{s} = \mathfrak{sl}$ and $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$.

Proof of Lemma 4.14 for $\mathfrak{s} = \mathfrak{sl}$. By Lemma 4.12 we can assume that $X - \lambda \text{Id}_V$ is rank-reduced. Therefore the linear operator $X - \lambda \text{Id}_V : V \rightarrow V$ is conjugate to a direct sum

$$\bigoplus_{1 \leq i \leq \text{rk } X_s} A(t_i) \quad \bigoplus_{1 \leq j \leq \text{rk } X_{nn}} B \quad \bigoplus C,$$

where C is the zero operator, and $A(t), B$ are operators of the following forms:

$$A(t) := \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We denote the elements of the respective bases of the 2-dimensional subspaces of V corresponding to $A(t_i)$ and B by $a_1[i], a_2[i]$ and $b_1[j], b_2[j]$, and the elements of the basis of the subspace corresponding to C by $c[k]$. We can further assume that

$$W := \text{span}\{a_1[i], b_1[j], c[k] \mid 1 \leq i \leq \text{rk } X_s, 1 \leq j \leq \text{rk } X_{nn}, 1 \leq k \leq d_W - r\}$$

and that the $\text{SL}(W)$ -invariant complement W' of W is the span of the remaining elements of the basis $\{a_1[i], a_2[i], b_1[j], b_2[j], c[k]\}_{i,j,k}$ of V . Then

$$\phi\left(\bigoplus_i \mathcal{O}[A(t_i)] \bigoplus_j \mathcal{O}[B] \bigoplus C\right) \subset \mathfrak{sl}\left(\bigoplus_{1 \leq i \leq \text{rk } X_s} \mathfrak{gl}\langle a_1[i], a_2[i] \rangle \bigoplus_{1 \leq j \leq \text{rk } X_{nn}} \mathfrak{sl}\langle b_1[j], b_2[j] \rangle\right).$$

Moreover, $\phi(\bigoplus_i \mathcal{O}[A(t_i)] \bigoplus_j \mathcal{O}[B] \bigoplus C)$ consists of all diagonal matrices of the form

$$\text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0), \quad \lambda_1 + \dots + \lambda_r = 0.$$

This follows from the obvious statement that the matrices in $\mathcal{O}[A(t)]$ for $t \neq 0$, and in $\mathcal{O}[B]$ can have arbitrary values in their left top corner. Therefore $\phi(\mathcal{O}[X])$ contains all semisimple elements of a given rank $r = \text{rk } X$. As such elements are dense in $\mathfrak{sl}(W)^{\leq r}$, $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{sl}(W)^{\leq r}$. \square

In the rest of this section $\mathfrak{s} = \mathfrak{so}$, \mathfrak{sp} . To prove Lemma 4.14 in this case, we need some preliminary lemmas.

Lemma 4.15. Let $X \in \mathfrak{s}(V)$. Assume that $2\text{rk}(X - \lambda \text{Id}_V) < d_V$ for some $\lambda \in \mathbb{F}$. Then $\lambda = 0$.

Proof. Obviously, $\text{rk}(X - \lambda \text{Id}_V) + d_V(\lambda) \geq d_V$ (recall that $d_V(\lambda) = \dim V_\lambda^X$). Therefore $2\text{rk}(X - \lambda \text{Id}_V) < d_V$ implies that $2d_V(\lambda) > d_V$.

However, for \mathfrak{so} , \mathfrak{sp} , we have $d_V(\lambda) = d_V(-\lambda)$. Therefore $\lambda \neq 0$ would mean that $\lambda \neq -\lambda$ and $\dim V_\lambda^X + \dim V_{-\lambda}^X > d_V$, which is false. Hence $\lambda = 0$. \square

Fix $r \in \mathbb{Z}_{\geq 0}$. If r is even we denote by $\mathfrak{s}(V)_{reg}^r$ the set of semisimple elements of $\mathfrak{s}(V)$ which have rank r . If r is odd we denote by $\mathfrak{sp}(V)_{reg}^r$ the set of elements X of $\mathfrak{sp}(V)$ such that

$$\text{rk } X_s = r - 1, \quad X_{ns} = 0, \quad \text{rk } X_{nn} = 1.$$

In particular, X_{nn} is nilpotent and has a single non-zero Jordan block of size 2×2 .

Lemma 4.16. a) Assume d_V is even. Then $\overline{\mathfrak{sp}(V)_{reg}^{d_V-1}} = \mathfrak{sp}(V)^{\leq (d_V-1)}$.
b) Assume d_V is odd. Then $\overline{\mathfrak{so}(V)_{reg}^{d_V-1}} = \mathfrak{so}(V)^{\leq (d_V-1)}$.

Proof. Exercise in linear algebra. \square

In connection with the following two lemmas, note that $\text{rk } X$ is even for any $X \in \mathfrak{so}(V)$. Recall also that on $\mathfrak{s}(V)$ there is the following partial order: X_1 is *lower* than X_2 if $\mathcal{O}[X_1] \subset \mathcal{O}[X_2]$. This order can be described explicitly in terms of Young diagrams (or equivalently, in terms of the Jordan normal forms of X_1, X_2), see [CM].

Lemma 4.17. Let r be an odd integer such that $1 \leq r \leq d_V$; assume that d_V is even. Then $\mathfrak{sp}(V)^{\leq r} = \overline{\mathfrak{sp}(V)_{reg}^r}$.

Proof. By definition $\mathfrak{sp}(V)_{reg}^r \subset \mathfrak{sp}(V)^{\leq r}$. Therefore is sufficient to prove that any element X of rank at most r lies in $\overline{\mathfrak{sp}(V)_{reg}^r}$. Fix $X \in \mathfrak{sp}(V)^{\leq r}$. There exists an X -stable orthogonal decomposition $V_s \oplus V_n = V$ such that $X|_{V_n}$ is nilpotent and $X|_{V_s}$ is given (in an appropriate basis) by a matrix of type

$$A := \begin{pmatrix} F & 0 \\ 0 & -F^t \end{pmatrix}.$$

Set $r_s := \text{rk } X|_{V_s}$, $r_n := \text{rk } X|_{V_n}$. Then r_s and $d_{V_s} := \dim V_s$ are even, and $r_s + r_n = r$.

To prove that $X \in \overline{\mathfrak{sp}(V)_{reg}^r}$ it suffices to prove that $X|_{V_s} \in \mathfrak{sp}(V)_{reg}^{r_s}$ and $X|_{V_n} \in \overline{\mathfrak{sp}(V_n)_{reg}^{r_n}}$. First of all we check that $X|_{V_s} \in \mathfrak{sp}(V_s)_{reg}^{r_s}$. The matrices of type A form a subalgebra isomorphic to \mathfrak{gl}_m for $m = \frac{d_{V_s}}{2}$. As $X|_{V_s} \in \mathfrak{gl}_m^{\leq \frac{r_s}{2}}$, we have $X|_{V_s} \in \overline{(\mathfrak{gl}_m)_{ss}^{\frac{r_s}{2}}}$. Therefore $X|_{V_s} \in \overline{\mathfrak{sp}(V_s)_{reg}^{r_s}}$.

Set $X' := X|_{V_n}$. It remains to show that $X' \in \overline{\mathfrak{sp}(V_n)_{reg}^{r_n}}$ (note that $\text{rk } X' = r_n$).

Let $X_{max} \in \mathfrak{sp}(V_n)$ be a nilpotent element of rank r_n with a single non-zero Jordan block of size $(r_n + 1) \times (r_n + 1)$. Such an element of $\mathfrak{sp}(V_n)$ exists by [CM], and moreover, $X' \in \overline{\mathcal{O}[X_{max}]}$.

It suffices to prove that $X_{max} \in \overline{\mathfrak{sp}(V_n)_{reg}^{r_n}}$. Using the explicit description of the coadjoint orbits of a symplectic group given in [CM] one can check that there exists an X_{max} -stable orthogonal decomposition $V'_n \oplus V''_n = V$ such that

$$X_{max}|_{V''_n} = 0, \quad d_{V'_n} = \text{rk } X_{max} + 1.$$

By Lemma 4.16, $X_{max}|_{V'_n} \in \mathfrak{sp}(V'_n)_{reg}^{r_n}$, and thus $X_{max} \in \overline{\mathfrak{sp}(V_n)_{reg}^{r_n}}$. \square

Lemma 4.18. Let r be an even integer such that $2 \leq r < d_V$. Then

$$\mathfrak{s}(V)^{\leq r} = \overline{\mathfrak{s}(V)_{reg}^r}.$$

Proof. By definition, $\mathfrak{s}(V)_{reg}^r \subset \mathfrak{s}(V)^{\leq r}$. Therefore it suffices to prove that any element X of rank at most r lies in $\overline{\mathfrak{s}(V)_{reg}^r}$. For $\mathfrak{s} = \mathfrak{so}$ the arguments are the same as in the proof of Lemma 4.17. This applies also to the case $\mathfrak{s} = \mathfrak{sp}$ if $\text{rk } X$ is odd. Thus in the remainder of the proof we can assume that $\mathfrak{s} = \mathfrak{sp}$ and $\text{rk } X$ is even.

Let $X_{max} \in \mathfrak{sp}(V_n)$ be a nilpotent element of rank r_n with two non-zero Jordan block of sizes $r_n \times r_n$ and 2×2 . Such an element of $\mathfrak{sp}(V_n)$ exists by [CM], and moreover, $X' \in \overline{\mathcal{O}[X_{max}]}$.

It suffices to prove that $X_{max} \in \overline{\mathfrak{sp}(V)_{reg}^r}$. There exists an X_{max} -stable orthogonal decomposition $V'_n \oplus V''_n \oplus V'''_n = V$ such that

$$d_{V''_n} = 2, \quad d_{V'_n} = \text{rk } X_{max}, \quad \text{and } X|_{V'''_n} = 0.$$

Since $X|_{V'_n} \in \overline{\mathfrak{sp}(V'_n)_{reg}^{d_{V'_n}-1}}$, $X|_{V'_n \oplus V''_n}$ lies in the closure in $\mathfrak{sp}(V'_n \oplus V''_n)$ of the set of elements $Y \in \mathfrak{sp}(V'_n \oplus V''_n)^{\leq r}$ such that

$$\text{rk } Y_{ss} = \text{rk } X_{max} - 2, \quad \text{rk } Y_{nn} = 2.$$

As $Y_{nn} \in \overline{\mathfrak{sp}(\mathbb{F}^4)_{reg}^2}$, we have $X_{max} \in \overline{\mathfrak{sp}(V)_{reg}^r}$. \square

Proof of Lemma 4.14 for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$. By Lemma 4.15, we have $\lambda = 0$, and therefore $\text{rk } X = r$. By Lemma 4.12 we can assume that X is rank-reduced. Note that the standard \mathfrak{gl}_2 -subalgebra of $\mathfrak{s}_4 \cong \mathfrak{so}_4, \mathfrak{sp}_4$ given by matrices

$$\begin{pmatrix} F & 0 \\ 0 & -F^t \end{pmatrix}, \quad F \in \mathfrak{gl}_2$$

in an appropriate basis.

Assume that $\text{rk } X$ is even. Then X is conjugate (via $S(V)$) to a direct sum

$$\bigoplus_{1 \leq i \leq \text{rk } X_s} A(t_i) \quad \bigoplus_{1 \leq j \leq \text{rk } X_{nn}} B \oplus C$$

where C is the zero-operator, and $A(t), B$ are of the form

$$A(t) := \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let $a_1[i], a_2[i], a_3[i], a_4[i]$ and $b_1[j], b_2[j], b_3[j], b_4[j]$ denote the elements of the basis of the 4-dimensional subspaces of V corresponding to $A(t_i)$ and B respectively, and let $c[k]$ be the elements of the basis of the subspace corresponding to C . The restriction of the fixed form to

$$E := \text{span}\{a_1[i], a_3[i], b_1[j], b_3[i], c[k] \mid 1 \leq i \leq \text{rk} X_s, 1 \leq j \leq \text{rk} X_{nn}, 1 \leq k \leq d_W - 2r\}$$

is nondegenerate and therefore we can assume that $W = E$. The orthogonal complement W^\perp to W is the span of the remaining elements in the above basis of V . Then

$$\phi\left(\bigoplus_i \mathcal{O}[A(t_i)] \bigoplus_j \mathcal{O}[B] \bigoplus C\right) \subset \bigoplus_{1 \leq i \leq \text{rk} X_s} \mathfrak{gl}_2 \bigoplus_{1 \leq j \leq \text{rk} X_{nn}} \mathfrak{gl}_2 \subset \mathfrak{s}(V),$$

and we claim that $\phi(\bigoplus_i \mathcal{O}[A(t_i)] \bigoplus_j \mathcal{O}[B] \bigoplus C)$ contains all diagonal matrices of the form

$$\text{diag}(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \dots, \lambda_{\text{rk} X}, -\lambda_{\text{rk} X}, 0, \dots, 0).$$

This follows from the obvious statement that matrices in $\mathcal{O}[A(t)]$ for $t \neq 0$ and in $\mathcal{O}[B]$ can have arbitrary values in their left top 2×2 -corner for $\mathfrak{s} = \mathfrak{sp}$ and arbitrary antisymmetric values for $\mathfrak{s} = \mathfrak{so}$.

Therefore $\phi(\mathcal{O}[X])$ contains all semisimple elements of rank $r = \text{rk} X$. As such elements are dense in $\mathfrak{s}(W)^{\leq r}$ for r even, $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{s}(W)^{\leq r}$.

It remains to consider the case when $\text{rk} X$ is odd. Here $\mathfrak{s} = \mathfrak{sp}$. Arguments similar to those above show that $\phi(\mathcal{O}[X])$ contains all elements of $\mathfrak{sp}(W)$ which has the form

$$\text{diag}(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \dots, \lambda_{\frac{r-1}{2}}, -\lambda_{\frac{r-1}{2}}, 0, \dots, 0) \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in an appropriate basis. Then $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{sp}(W)^{\leq r}$ by Lemma 4.17. \square

5. PROOF OF THEOREMS 3.1 AND 3.3

In the rest of the paper we assume that \mathfrak{g}_∞ is a locally simple Lie algebra which may be finitary (i.e. isomorphic to $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$), diagonal, or non-diagonal. Let $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$ for a fixed sequence of embeddings (1). In the special case of $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ we assume in addition that the simple Lie algebras \mathfrak{g}_n satisfy the assumptions from Section 3. Let G_n be the adjoint group of \mathfrak{g}_n . Recall that, if $J \subset \mathbf{S}(\mathfrak{g})$ is a Poisson ideal of locally infinite codimension, the intersections $J_n := J \cap \mathbf{S}(\mathfrak{g}_n)$ determine proper G_n -stable closed subvarieties $\text{Var}(J_n) \subset \mathfrak{g}_n^*$ which form an inverse system

$$\dots \rightarrow \text{Var}(J_n) \xrightarrow{p_n} \text{Var}(J_{n-1}) \rightarrow \dots$$

under the natural projections $\mathfrak{g}_n^* \rightarrow \mathfrak{g}_{n-1}^*$. Moreover, $\text{Var}(J_{n-1}) = \overline{\text{pr}_n \text{Var}(J_n)}$ where the closure is taken in \mathfrak{g}_{n-1}^* .

Let \mathfrak{k}_1 be a simple finite-dimensional Lie algebra and $\mathfrak{k}_2 \subset \mathfrak{k}_1$ be a simple finite-dimensional subalgebra. The restriction of the Cartan-Killing of \mathfrak{k}_1 to \mathfrak{k}_2 is proportional to the Cartan-Killing form of \mathfrak{k}_2 with coefficient which we denote $I_{\mathfrak{k}_2}^{\mathfrak{k}_1}$. This coefficient, known as the *Dynkin index*, is multiplicative: if $\mathfrak{k}_3 \subset \mathfrak{k}_2 \subset \mathfrak{k}_1$ is a chain of inclusions, then

$$I_{\mathfrak{k}_3}^{\mathfrak{k}_1} = I_{\mathfrak{k}_3}^{\mathfrak{k}_2} I_{\mathfrak{k}_2}^{\mathfrak{k}_1}.$$

The Dynkin index is always a positive integer [Dy]. Moreover, if $\mathfrak{k}_2 \subset \mathfrak{k}_1$ are classical simple Lie algebras of the same type and of rank at least 5, then $I_{\mathfrak{k}_2}^{\mathfrak{k}_1} = 1$ if and only if the natural module of \mathfrak{k}_1 decomposes over \mathfrak{k}_2 as a natural plus a trivial module [DP, Proposition 2.3].

Proof of Theorem 3.1. Let J be a non-zero Poisson ideal of locally infinite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$. Without loss of generality we may assume that J is a radical ideal, as the radical of a Poisson of locally infinite codimension ideal in $\mathbf{S}(\mathfrak{g}_\infty)$ is again Poisson and of locally infinite codimension.

Fix n so that $J_n = J \cap \mathbf{S}(\mathfrak{g}_n)$ is non-zero and of infinite codimension in $\mathbf{S}(\mathfrak{g}_n)$. The image of any G_{n+m} -orbit in $\text{Var}(J_{n+m})$ under the morphism $\text{Var}(J_{n+m}) \rightarrow \mathfrak{g}_n^*$ is not dense in \mathfrak{g}_n^* as it lies in the proper closed subvariety $\text{Var}(J_n) \subset \mathfrak{g}_n^*$. Therefore Proposition 4.2 implies that $\dim(\mathfrak{g}_n \cdot V_{n+m})$ is bounded as a function on m . Hence the number of non-trivial simple \mathfrak{g}_n -constituents in V_{n+m} and their dimensions are simultaneously bounded. As a consequence, $I_{\mathfrak{g}_n}^{\mathfrak{g}_{n+m}} = 1$ is bounded as a function of $m > 0$. Therefore there exists $N > 0$ such that $I_{\mathfrak{g}_n}^{\mathfrak{g}_{n+1}} = 1$ for all $n > N$. Now [DP, Corollary 2.4] implies $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ or \mathfrak{sp}_∞ . \square

Proof of Theorem 3.3. If J is of locally finite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$, then

$$\text{Var}(J) = 0 = \mathfrak{s}_\infty^{\leq 0}.$$

Therefore, in the rest of the proof we can assume that J is non-zero and of locally infinite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$.

Fix n so that $J_n = J \cap \mathbf{S}(\mathfrak{g}_n)$ is non-zero and of infinite codimension in $\mathbf{S}(\mathfrak{g}_n)$. Since for any non-zero $X \in \text{Var}(J_{m+n})$ the image in \mathfrak{g}_n^* of the G_{m+n} -orbit $\mathcal{O}[X] \subset \mathfrak{g}_{m+n}^*$ is not dense in \mathfrak{g}_n^* , Lemma 4.13 implies that $\text{Var}(J_{m+n}) \subset \mathfrak{g}_{m+n}^{\leq r'_n}$ for some r'_n which depends on n only. Let r_n be the minimal such r'_n . By Lemma 4.13, $r_n < n$.

The inequality $r_n < n$ allows us to apply Lemma 4.14 when $m > n$. It implies that the image in $\mathfrak{g}_n^{\leq r_n}$ of $\mathcal{O}[X] \subset \mathfrak{g}_{m+n}^*$ is dense in $\mathfrak{g}_n^{\leq r_n}$ for any $X \in \text{Var}(J_{m+n})$ with $\text{rk } X = r_n$. Furthermore, by definition, $r_n \leq r_{n+m}$. Lemma 4.7 implies that $r_{m+n} \geq r_n$, and therefore $r_n = r_{m+n}$. Set $r = r_n$. Then $J = J^{\leq r}$. \square

6. SOME COROLLARIES

Corollary 3.2 implies that if \mathfrak{g}_∞ is a locally simple Lie algebra which is not finitary, then any ideal in $\mathbf{U}(\mathfrak{g}_\infty)$ is of locally finite codimension. Furthermore, a result of A. Zhilinskii [Zh2] claims that $\mathbf{U}(\mathfrak{g}_\infty)$ admits an ideal of locally finite codimension which is not the augmentation ideal if and only if \mathfrak{g}_∞ is diagonal. These two statements yield the following corollary.

Corollary 6.1 (Baranov's Conjecture). Let \mathfrak{g}_∞ be any locally simple Lie algebra. Then the augmentation ideal is the only non-zero ideal in $\mathbf{U}(\mathfrak{g}_\infty)$ if and only if \mathfrak{g}_∞ is not diagonal.

Furthermore, Theorems 3.1, 3.3 imply the following.

Corollary 6.2. Let \mathfrak{g}_∞ be any locally simple Lie algebra and $I \subset \mathbf{U}(\mathfrak{g}_\infty)$ be an ideal. Then

- a) $\text{Var}(I) \neq 0$ implies that $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$;
- b) if $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ and $I \subset \mathbf{U}(\mathfrak{g}_\infty)$ is an ideal of locally infinite codimension, then $\text{Var}(I) = \mathfrak{s}_\infty^{\leq r}$ for some $r \in \mathbb{Z}_{\geq 1}$.

A. Zhilinskii [Zh3] (see also [Zh1]) has given a description of all ideals of locally finite codimension in $U(\mathfrak{g}_\infty)$ for an arbitrary locally simple Lie algebra \mathfrak{g}_∞ . Therefore the problem of describing all ideals in $U(\mathfrak{g}_\infty)$ gets reduced to the problem of describing all ideals of locally infinite codimension in $U(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. In the following two sections we will show in particular that all proj-varieties $\mathfrak{s}^{\leq r}$ arise as associated “varieties” of ideals respectively of $U(\mathfrak{sl}_\infty), U(\mathfrak{so}_\infty), U(\mathfrak{sp}_\infty)$.

7. COHERENT LOCAL SYSTEMS OF MODULES AND A CLASSIFICATION OF PRIME INTEGRABLE IDEALS OF $U(\mathfrak{g}_\infty)$

In this section we review some published and unpublished results of A. Zhilinskii and draw corollaries.

Definition 7.1. An ideal $I \subset U(\mathfrak{g}_\infty)$ is *integrable* if for any finitely generated subalgebra $U' \subset U(\mathfrak{g}_\infty)$, the ideal $I \cap U'$ in U' is an intersection of ideals of finite codimension of U' .

If a \mathfrak{g}_∞ -module M is *integrable*, i.e. $\dim U(\mathfrak{g}')m < \infty$ for any $m \in M$ and any finite-dimensional subalgebra $\mathfrak{g}' \subset \mathfrak{g}_\infty$, the annihilator of M in $U(\mathfrak{g}_\infty)$ is an integrable ideal. Note that an equivalent definition of an integrable $U(\mathfrak{g}_\infty)$ module is a left $U(\mathfrak{g}_\infty)$ -module M for which $\dim(U'm) < \infty$ for any $m \in M$ and any finitely generated subalgebra $U' \subset U(\mathfrak{g}_\infty)$. Integrable ideals in $U(\mathfrak{g}_\infty)$ are described as annihilators of coherent local systems of finite-dimensional \mathfrak{g}_n -modules as introduced by A. Zhilinskii in [Zh2]. We discuss this topic below.

7.1. Integrable ideals and coherent local systems.

Definition 7.2. A *coherent local system of modules* (further shortened as *c.l.s.*) for $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$ is a collection of sets

$$\{Q_n\}_{n \in \mathbb{Z}_{\geq 1}} \subset \prod_{n \in \mathbb{Z}_{\geq 1}} \text{Irr } \mathfrak{g}_n$$

such that $Q_m = \langle Q_n \rangle_m$ for any $n > m$.

If Q is a c.l.s., then $\cap_{z \in Q_m} \text{Ann}_{U(\mathfrak{g}_m)} z \subset \cap_{z \in Q_n} \text{Ann}_{U(\mathfrak{g}_n)} z$ for any $n > m$. Therefore $\cup_m (\cap_{z \in Q_m} \text{Ann}_{U(\mathfrak{g}_m)} z)$ is an ideal of $U(\mathfrak{g})$; we denote it by $I(Q)$. Note that $I(Q)$ is integrable.

A. Zhilinskii [Zh1], [Zh2], [Zh3] has classified c.l.s. for any locally simple Lie algebra \mathfrak{g}_∞ . Below we show how this classification leads to a description of integrable ideals of $U(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

A c.l.s. Q is *irreducible* if $Q \neq Q' \cup Q''$ with $Q' \not\subset Q''$ and $Q'' \not\subset Q'$. Let Q be any c.l.s. and let $\mathcal{S}(Q)$ be the set of irreducible c.l.s. which are contained in Q . Then $\mathcal{S}(Q)$ has a finite subset of elements $Q(1), \dots, Q(r)$ which are maximal by inclusion, and $Q = \cup_r Q(r)$ [Zh1]; we call $Q(r)$ the *irreducible components* of Q . This makes apparent the analogy between c.l.s. and algebraic varieties.

Any integrable \mathfrak{g}_∞ -module M determines a c.l.s. $Q := \{Q_n\}_{n \in \mathbb{Z}_{\geq 0}}$, where

$$Q_n := \{z \in \text{Irr } \mathfrak{g}_n \mid \text{Hom}_{\mathfrak{g}_n}(z, M) \neq 0\}.$$

We denote this relation by $Q \leftarrow M$. We also recall that an integrable \mathfrak{g}_∞ -module M is *locally simple* if $M = \varinjlim M_n$ for a suitable chain $\dots \subset M_n \subset M_{n+1} \subset \dots$ of simple finite-dimensional \mathfrak{g}_n -submodules M_n of M .

Proposition 7.3 ([Zh1, Lemma 1.1.2]). If Q is an irreducible c.l.s., then $I(Q)$ is the annihilator of some locally simple integrable \mathfrak{g}_∞ -module. In particular, $I(Q)$ is primitive and hence prime.

Fix n . The set $\text{Irr}\mathfrak{g}_n$ is parametrized by the lattice Λ_i of integral dominant weights of \mathfrak{g}_n . Let z_1, z_2 be isomorphism classes of simple \mathfrak{g}_n -modules with respective highest weights λ_1, λ_2 . We denote by $z_1 z_2$ the isomorphism class of a simple module with highest weight $\lambda_1 + \lambda_2$. If $S_1, S_2 \subset \text{Irr}\mathfrak{g}_n$ we set

$$S_1 S_2 := \{z \in \text{Irr}\mathfrak{g}_n \mid z = z_1 z_2 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

If Q', Q'' are c.l.s., we denote by $Q'Q''$ the smallest c.l.s. such that $(Q')_i(Q'')_i \subset (Q'Q'')_i$. By definition, $Q'Q''$ is the *product* of Q' and Q'' . If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, then by [Zh1]

$$(Q')_n(Q'')_n = (Q'Q'')_n.$$

7.2. Zhilinskii's classification of c.l.s. In this subsection we reproduce A. Zhilinskii's classification of irreducible c.l.s. for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

In the rest of the paper $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ or \mathfrak{sp}_∞ and $\mathfrak{g}_n = \mathfrak{s}(V_n)$ is a sequence (1) of finite-dimensional simple Lie algebras satisfying the assumptions of Section 3. We set $V_\infty := \varinjlim V_n$ and $(V_\infty)_* := \varinjlim V_n^*$.

The following irreducible c.l.s. are by definition the *basic c.l.s.*:

$$\begin{aligned} &\text{for } \mathfrak{g}_\infty = \mathfrak{sl}_\infty : \mathcal{E} \leftarrow \Lambda^* V_\infty, \quad \mathcal{L}_p \leftarrow \Lambda^p V_\infty, \quad \mathcal{L}_p^\infty \leftarrow \mathbf{S}^*(V_\infty \otimes \mathbb{F}^p), \\ &\quad \mathcal{R}_q \leftarrow \Lambda^q (V_\infty)_*, \quad \mathcal{R}_q^\infty \leftarrow \mathbf{S}^*((V_\infty)_* \otimes \mathbb{F}^q), \quad \mathcal{E}^\infty \text{ (all modules);} \\ &\text{for } \mathfrak{g}_\infty = \mathfrak{sp}_\infty : \mathcal{E} \leftarrow \Lambda^* V_\infty, \quad \mathcal{L}_p \leftarrow \Lambda^p V_\infty, \quad \mathcal{L}_p^\infty \leftarrow \mathbf{S}^*(V_\infty \otimes \mathbb{F}^p), \\ &\quad \mathcal{E}^\infty \text{ (all modules);} \\ &\text{for } \mathfrak{g}_\infty = \mathfrak{so}_\infty : \mathcal{E} \leftarrow \Lambda^* V_\infty, \quad \mathcal{L}_p \leftarrow \Lambda^p V_\infty, \quad \mathcal{L}_p^\infty \leftarrow \mathbf{S}^*(V_\infty \otimes \mathbb{F}^p), \\ &\quad \mathcal{R} \text{ (spinor modules), } \mathcal{E}^\infty \text{ (all modules),} \end{aligned}$$

where $p, q \in \mathbb{Z}_{\geq 1}$.

Proposition 7.4 (Unique factorization property [Zh1]). Any irreducible c.l.s. can be expressed uniquely as a product as follows:

$$\begin{aligned} &(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m (\mathcal{R}_w^\infty \mathcal{R}_{w+1}^{z_{w+1}} \mathcal{R}_{w+2}^{z_{w+2}} \dots \mathcal{R}_n^{z_n}) && \text{for } \mathfrak{g}_\infty = \mathfrak{sl}_\infty, \quad (4) \\ &(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m \text{ or } (\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m \mathcal{R} && \text{for } \mathfrak{g}_\infty = \mathfrak{so}_\infty, \quad (5) \\ &(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m && \text{for } \mathfrak{g}_\infty = \mathfrak{sp}_\infty, \quad (6) \end{aligned}$$

where

$$\begin{aligned} &m, n, v, w \in \mathbb{Z}_{\geq 0}, \quad v, w \leq n, \\ &x_i, z_j \in \mathbb{Z}_{\geq 0} \text{ for } v+1 \leq i \leq n \text{ and } w+1 \leq j \leq n. \end{aligned}$$

Here, for $v = 0$, \mathcal{L}_v^∞ is assumed to be the identity (the c.l.s. consisting of the isomorphism class of the trivial 1-dimensional module at all levels), and for $w = 0$, \mathcal{R}_w^∞ is assumed to be the identity.

We say that an irreducible c.l.s. of \mathfrak{so}_∞ is of *integer type* if its expression does not contain \mathcal{R} , and of *semiinteger type* otherwise. Furthermore, we define a c.l.s. Q to be of *finite type* if the set Q_n is finite for all $n \geq 1$. It is easy to see that the ideal $I(Q)$ is of locally finite-codimension in $U(\mathfrak{g}_\infty)$ if and only if Q is of finite type.

If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, the irreducible c.l.s. of finite type form a free lattice (by means of product) generated by $\mathcal{E}, \mathcal{L}_p, \mathcal{R}_q$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ and by $\mathcal{E}, \mathcal{L}_p$ for $\mathfrak{g}_\infty = \mathfrak{sp}_\infty$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty$ the set of irreducible c.l.s. of finite type equals the union $\mathcal{N} \sqcup \mathcal{N}\mathcal{R}$, where \mathcal{N} is a free lattice generated by $\mathcal{E}, \mathcal{L}_p$, and $\mathcal{N}\mathcal{R} := \{\mathcal{N}\mathcal{R} \mid \mathcal{N} \in \mathcal{N}\}$ [Zh1].

7.3. Partial order by inclusion on c.l.s. To a c.l.s. Q for \mathfrak{sl}_∞ in the form (4) A. Zhilinskii assigns the following two non-increasing sequences of elements of $\mathbb{Z}_{\geq 0} \cup \{+\infty\}$

$$\{l_i := m + \Sigma_{j \geq i} x_j\}_i \text{ and } \{r_i := m + \Sigma_{j \geq i} z_j\}_i,$$

where it is assumed $l_1 = l_2 = \dots = l_v := +\infty =: r_1 = \dots = r_w$. Note that

$$\lim_{i \rightarrow \infty} l_i = \lim_{i \rightarrow \infty} r_i = m.$$

Similarly, to a c.l.s. for \mathfrak{sp}_∞ or \mathfrak{so}_∞ in the form (5) or (6) A. Zhilinskii assigns the non-increasing sequence

$$\{l_i := m + \Sigma_{j \geq i} x_i\}_i,$$

where $l_1 = \dots = l_v := +\infty$. Again $\lim_{i \rightarrow \infty} l_i = m$.

Zhilinskii establishes the following inclusion criterion [Zh1]. A c.l.s. Q for \mathfrak{sl}_∞ contains a c.l.s. Q' if and only if, for some $a, b \in \mathbb{Z}_{\geq 0}$, we have

$$a + b = m - m', l_i \geq l'_i + a, r_i \geq r'_i + b$$

for the corresponding sequences $\{l_i\}, \{r_i\}$ and $\{l'_i\}, \{r'_i\}$. If Q and Q' are c.l.s. of \mathfrak{sp}_∞ , then $Q' \subset Q$ if and only if

$$l_i \geq l'_i.$$

Finally, if Q and Q' are c.l.s. of \mathfrak{so}_∞ , then $Q' \subset Q$ if and only if Q and Q' have the same integer or semiinteger type, and

$$l_i \geq l'_i.$$

Corollary 7.5. If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, the only minimal c.l.s. is the trivial c.l.s. (i.e. the c.l.s. Q with Q_n being the isomorphism class of the 1-dimensional trivial \mathfrak{g}_n -module for all $n \geq 1$). If $\mathfrak{g}_\infty = \mathfrak{so}_\infty$, there are two minimal c.l.s.: the trivial one and \mathcal{R} .

Proof. Follows from the inclusion criterion of A. Zhilinskii. \square

7.4. Tensor product and ideals. Fix i . If $S_1, S_2 \subset \text{Irr } \mathfrak{g}_n$ we set

$$S_1 \otimes S_2 := \{z \in \text{Irr } \mathfrak{g}_n \mid \text{Hom}(z, z_1 \otimes z_2) \neq 0 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

Furthermore, it is clear that the tensor product of two c.l.s. is a well defined c.l.s.. One can check that

$$(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m (\mathcal{R}_w^\infty \mathcal{R}_{w+1}^{z_{w+1}} \mathcal{R}_{w+2}^{z_{w+2}} \dots \mathcal{R}_n^{z_n}) = \\ (\mathcal{L}_1^\infty)^{\otimes v} \otimes (\mathcal{R}_1^\infty)^{\otimes w} \otimes ((\mathcal{L}_1^{x_{v+1}} \mathcal{L}_2^{x_{v+2}} \dots \mathcal{L}_{n-v}^{x_n}) \mathcal{E}^m (\mathcal{R}_1^{z_{w+1}} \mathcal{R}_2^{z_{w+2}} \dots \mathcal{R}_{n-w}^{z_n}))$$

for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$, and

$$(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m = (\mathcal{L}_1^\infty)^{\otimes v} \otimes ((\mathcal{L}_1^{x_{v+1}} \mathcal{L}_2^{x_{v+2}} \dots \mathcal{L}_{n-v}^{x_n}) \mathcal{E}^m), \\ (\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m \mathcal{R} = (\mathcal{L}_1^\infty)^{\otimes v} \otimes ((\mathcal{L}_1^{x_{v+1}} \mathcal{L}_2^{x_{v+2}} \dots \mathcal{L}_{n-v}^{x_n}) \mathcal{E}^m \mathcal{R}) \quad (7)$$

for $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, where $(\mathcal{L}_1^\infty)^{\otimes v} := \mathcal{L}_1^\infty \otimes \mathcal{L}_1^\infty \otimes \dots \otimes \mathcal{L}_1^\infty$ (v times). Note that formula (7) applies to \mathfrak{so}_∞ -case only.

The above formulas yield a different parametrization of the irreducible c.l.s.: for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ the irreducible c.l.s. are parametrized by triples (v, w, Q_f) , where $v, w \in \mathbb{Z}_{\geq 0}$ and Q_f is an irreducible c.l.s. of finite type, and for $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ the irreducible c.l.s. are parametrized by pairs (v, Q_f) , where $v \in \mathbb{Z}_{\geq 0}$ and Q_f is an irreducible c.l.s. of finite type.

For $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ we set

$$\text{cls}(v, w, Q_f) := (\mathcal{L}_1^\infty)^{\otimes v} \otimes (\mathcal{R}_1^\infty)^{\otimes w} \otimes Q_f$$

and

$$I(v, w, Q_f) := I(\text{cls}(v, w, Q_f)).$$

However, it is easy to check that the annihilators of \mathcal{L}_1^∞ and \mathcal{R}_1^∞ in $U(\mathfrak{sl}_\infty)$ coincide. Therefore

$$I(v, w, Q_f) = I(v + w, 0, Q_f).$$

In what follows we call an irreducible c.l.s. for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ of the form $\text{cls}(v, 0, Q_f)$ a *left irreducible* c.l.s. An arbitrary *left* c.l.s. is defined as a finite union of left irreducible c.l.s. We denote the left irreducible c.l.s. for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ by $\text{cls}(v, Q_f)$. We also set $I(v, Q_f) := I(v, 0, Q_f)$.

Note that for $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ an arbitrary irreducible c.l.s. corresponds to a pair (v, Q_f) . Therefore the notations $\text{cls}(v, Q_f)$ and $I(v, Q_f)$ make sense in all three cases: $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

7.5. Classification of integrable ideals in $U(\mathfrak{g}_\infty)$. Recall that $\mathfrak{g}_n = \mathfrak{sl}_{n+1}$ for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$. In this case the space of weights of \mathfrak{g}_n is identified with the set of $(n+1)$ -tuples $(\lambda_1, \dots, \lambda_{n+1})$, $\lambda_i \in \mathbb{F}$, up to the equivalence relation

$$(\lambda_1, \dots, \lambda_{n+1}) \sim (\lambda_1 + k, \dots, \lambda_{n+1} + k)$$

for $k \in \mathbb{F}$. The lattice of integral dominant weights is identified with the set of weights such that $\lambda_{i+1} - \lambda_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n-1$.

For $\mathfrak{g}_\infty \cong \mathfrak{sp}_\infty$, the space of weights of $\mathfrak{g}_n \cong \mathfrak{sp}_{2n}$ is identified with the set of n -tuples $(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{F}$. The lattice of integral dominant weights is identified with the set of weights such that $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_{i+1} \geq \lambda_i$ for $1 \leq i \leq n-1$.

For $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty$, we assume that $\mathfrak{g}_n \cong \mathfrak{so}_{2n}$. The space weights of $\mathfrak{g}_n = \mathfrak{so}_{2n}$ is identified with the set of n -tuples $(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{F}$. The lattice of integral dominant weights is identified with the set of weights such that $\lambda_i \in \frac{1}{2}\mathbb{Z}$, $\lambda_{i+1} - \lambda_i \in \mathbb{Z}_{\leq 0}$ for $1 \leq i \leq n-1$ and $\lambda_{n-1} \geq |\lambda_n|$.

Let \mathcal{W}_n denote the Weyl group of \mathfrak{g}_n , and let ρ_n be the half-sum of positive roots of \mathfrak{g}_n . The set of radical ideals of $Z_{U(\mathfrak{g}_n)}$ is identified with the set of Zariski-closed \mathcal{W}_n -stable subsets of the space of weights of \mathfrak{g}_n .

Let Q be a c.l.s. For any $n \in \mathbb{Z}_{\geq 1}$ we denote by $\overline{Q_n}$ the Zariski-closure of the set of highest weights of the isomorphism classes of simple \mathfrak{g}_n -modules from Q_n in the space of weights of \mathfrak{g}_n .

Lemma 7.6. a) For any $v \in \mathbb{Z}_{\geq 0}$ and any c.l.s. Q_f of finite type, the following conditions are equivalent:

- (i) $(v, Q_f) = (v', Q'_f)$,
- (ii) $\mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n}) = \mathcal{W}_n(\rho_n + \overline{\text{cls}(v', Q'_f)_n})$ for all $n \in \mathbb{Z}_{\geq 1}$.

b) Let $z \in \text{Irr } \mathfrak{g}_n$ and let λ_z be the highest weight of a representative of z . For $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, we have

$$\lambda_z + \rho_n \in \mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n})$$

if and only if $z \in \text{cls}(v, Q_f)_n$. For $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$, we have

$$\lambda_z + \rho_n \in \mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n})$$

if and only if there exist $v', v'' \in \mathbb{Z}_{\geq 0}$ such that $v' + v'' = v$ and $z \in \text{cls}(v', v'', Q_f)_n$.

c) Let $v \in \mathbb{Z}_{\geq 0}$ and Q_f be a c.l.s. for \mathfrak{g}_∞ . Then, for $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ we have $Q(I(v, Q_f)) = \text{cls}(v, Q_f)$. For $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$ we have

$$Q(I(v, Q_f)) = \cup_{v'+v''=v} \text{cls}(v', v'', Q_f).$$

Proof. Part a) follows from the following explicit description of the closure $\overline{\text{cls}(v, Q_f)_{n+v}}$: if Q_f is an irreducible c.l.s. of finite type, then $(\lambda_1, \dots, \lambda_{n+v}) \in \overline{\text{cls}(v, Q_f)_{n+v}}$ if and only if the simple \mathfrak{g}_n -module with highest weight $(\lambda_{v+1}, \dots, \lambda_{n+v})$ lies in $(Q_f)_n$.

Part b) is a straightforward corollary of part a). We proceed to c).

We fix v, Q_f, n . Let $z \in Q(I(v, Q_f))_n$ and λ_z be the highest weight of z . By definition, $I \cap U(\mathfrak{g}_n) \subset \text{Ann}_{U(\mathfrak{g}_n)} z$. Then $I \cap Z(U(\mathfrak{g}_n)) \subset (\text{Ann}_{U(\mathfrak{g}_n)} z) \cap Z(U(\mathfrak{g}_n))$. This inclusion is equivalent to the condition

$$\lambda_z + \rho_n \in \mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n}).$$

Therefore $z \in \text{cls}(v, Q_f)_n$ for $\mathfrak{g}_\infty \cong \mathfrak{sp}_\infty, \mathfrak{so}_\infty$, and $z \in \cup_{v'+v''=v} \text{cls}(v', v'', Q_f)_n$ for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$ by b).

Hence $Q(I(v, Q_f)) = \text{cls}(v, Q_f)$ for $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

Assume that $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$. Then $I(v', v'', Q_f) = I(v' + v'', Q_f)$ and thus $\text{cls}(v', v'', Q_f) \subset Q(I(v' + v'', Q_f))$. Hence $Q(I(v, Q_f)) \cup_{v'+v''=v} \text{cls}(v', v'', Q_f)$. \square

For any ideal $I \subset U(\mathfrak{g}_\infty)$, set

$$Q(I)_n := \{z \in \text{Irr } \mathfrak{g}_n \mid I \cap U(\mathfrak{g}_n) \subset \text{Ann}_{U(\mathfrak{g}_n)} z\}$$

and note that $Q(I)$ is a well-defined c.l.s.. Note that if I is integrable, then I is the annihilator of $Q(I)$.

Proposition 7.7. Assume $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. An integrable ideal of $U(\mathfrak{g}_\infty)$ is prime if and only if it is primitive.²

Proof. Let I be a prime integrable ideal of $U(\mathfrak{g}_\infty)$. Then I is the annihilator of $Q(I)$. Let Q_1, \dots, Q_s be the irreducible components of $Q(I)$. We have

$$I = I(Q(I)) \subset \cap_{i \leq s} I(Q_i)$$

and

$$I(Q_1)I(Q_2)\dots I(Q_s) \subset I(Q(I)) = I.$$

Therefore $I = I(Q(I))$ coincides with $I(Q_i)$ for some irreducible c.l.s. Q_i . Thus the ideal $I(Q_i)$ is primitive by Proposition 7.3. On the other hand, any primitive ideal is prime. \square

For $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$, we denote by $Q_l(I)$ the union of all left irreducible components of $Q(I)$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, we put $Q_l(I) = Q(I)$.

Theorem 7.8. a) The maps

$$\begin{aligned} I &\mapsto Q_l(I), \\ Q &\mapsto I(Q) \end{aligned}$$

are mutually inverse bijections between the set of prime integrable ideals $I \subset U(\mathfrak{g}_\infty)$ and the set of irreducible left c.l.s..

b) If $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, the maps in a) extend to mutually inverse anti-isomorphisms

$$I \mapsto Q(I), \tag{8}$$

$$Q \mapsto I(Q) \tag{9}$$

between the lattice of integrable ideals in $U(\mathfrak{g}_\infty)$ and the lattice of c.l.s. for \mathfrak{g}_∞ ³.

c) If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$, any integrable ideal of $U(\mathfrak{sl}_\infty)$ equals $I(Q)$ for some left c.l.s. Q for \mathfrak{g}_∞ .

²The analogous statement is false for \mathfrak{g}_n . For instance, $I(\mathcal{L}_1) \cap U(\mathfrak{g}_n), n \geq 2$, is an integrable prime ideal of $U(\mathfrak{g}_n)$ which is not primitive.

³Contrary to our conventions from Section 2, here we consider $U(\mathfrak{g}_\infty)$ as an integrable ideal and $\{0\}$ as c.l.s.

Proof. Let I be a prime integrable ideal of $U(\mathfrak{g}_\infty)$. By definition, I is the annihilator of $Q(I)$. Let Q_1, \dots, Q_s be the irreducible components of $Q(I)$. Then

$$I = I(Q(I)) \subset \cap_{i \leq s} I(Q_i)$$

and

$$I(Q_1)I(Q_2)\dots I(Q_s) \subset I(Q(I)) = I.$$

Therefore $I = I(Q(I))$ coincides with $I(Q_i)$ for some irreducible c.l.s. Q_i . Hence, by Subsection 7.4, $I = I(v, Q_f)$ for some $v \in \mathbb{Z}_{\geq 0}$ and some c.l.s. of finite type Q_f .

To prove a) it remains to check that the annihilators of the c.l.s. corresponding to distinct pairs (v, Q_f) and (v', Q'_f) are distinct. For $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ we have to prove that $I(v, 0, Q_f) = I(v', 0, Q'_f)$ if and only if $v = v'$ and $Q_f = Q'_f$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, we have to check that $I(v, Q_f) = I(v', Q'_f)$ if and only if $v = v'$ and $Q_f = Q'_f$. Both statements follow from Lemma 7.6. Hence, a) is proved.

To prove b) we assume that $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. Let Q be any c.l.s.. Then $Q = \cup_{i \leq s} Q_i$ for some irreducible c.l.s. Q_i , and $I(Q) = \cap_{i \leq s} I(Q_i)$. On the other hand, $Q(I(\cap_{i \leq s} I(Q_i))) = Q(\cap_{i \leq s} I(Q_i))$ by definition, and

$$Q(\cap_{i \leq s} I(Q_i)) = \cup_{i \leq s} (Q(I(Q_i))) = \cup_{i \leq s} Q_i = Q$$

by a). Therefore the maps (8) and (9) are mutually inverse. In addition, it is now obvious that both maps are anti-homomorphisms of lattices. This proves b).

To prove c) we assume that $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$. Let I be an integrable ideal of $U(\mathfrak{sl}_\infty)$. Then $I = \cap_{i \leq s} I_i$, where I_i are prime integrable ideals of $U(\mathfrak{sl}_\infty)$. For any $i \leq s$, $I_i = I(v_i, (Q_f)_i)$ for some $(v_i, (Q_f)_i)$ as in a). In particular, I_i is the annihilator of a left local system $Q_i := (v_i, 0, (Q_f)_i)$. Then I is the annihilator of $\cup_{i \leq s} Q_i$. This proves c). \square

Note that for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$ the annihilators of \mathcal{L}_2^∞ and $(\mathcal{L}_2^\infty) \cup (\mathcal{L}_1^\infty \mathcal{R}_1)$ coincide. This shows in particular that the one-to-one correspondence between left irreducible c.l.s. of \mathfrak{g}_∞ and prime integrable ideals of $U(\mathfrak{g}_\infty)$ can not be extended to an anti-isomorphism between the corresponding lattices.

Nevertheless, Theorem 7.8 c) provides a certain description of general integrable ideal of $U(\mathfrak{sl}_\infty)$: it yields a surjection from the set of c.l.s. of \mathfrak{sl}_∞ to the set of integrable ideals of $U(\mathfrak{sl}_\infty)$. Two c.l.s. Q_1, Q_2 determine the same integrable ideal $I(Q_1) = I(Q_2)$ if and only if $Q(I(Q_1)) = Q(I(Q_2))$. For any irreducible c.l.s. Q , the c.l.s. $Q(I(Q))$ is described by Lemma 7.6 c). If Q is a reducible c.l.s., i.e. $Q = \cup_{j \leq s} Q_j$ for some irreducible c.l.s. Q_1, \dots, Q_j , then $Q(I(Q)) = \cup_{j \leq s} Q(I(Q_j))$. This allows in principle to check when $Q(I(Q_1)) = Q(I(Q_2))$.

Furthermore, let's point out that Theorem 7.8, together with A. Zhilinskii's result [Zh1], [Zh3] that any coherent local system is the union of finitely many coherent local systems, implies that the lattice of integrable ideals of $U(\mathfrak{g}_\infty)$ satisfies the ascending chain condition. This has already been stated in [Zh1] and [Zh3]. We would like to think of this result as of "relative Nötherianity" of the algebra $U(\mathfrak{g}_\infty)$. We don't know whether $U(\mathfrak{g}_\infty)$ is two-sided Nötherian.

Theorem 7.8 and Corollary 7.5 imply immediately that if $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, the augmentation ideal is the unique integrable maximal ideal of $U(\mathfrak{g}_\infty)$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty$ there are two integrable maximal ideals: the augmentation ideal and the "spinor ideal" $I(\mathcal{R})$.

Corollary 7.9. The algebras $U(\mathfrak{sl}_\infty)$, $U(\mathfrak{so}_\infty)$ and $U(\mathfrak{sp}_\infty)$ are pairwise nonisomorphic.

Proof. The algebras $U(\mathfrak{sl}_\infty)$, $U(\mathfrak{sp}_\infty)$ have each a unique integrable maximal ideal, while the algebra $U(\mathfrak{so}_\infty)$ has two integrable maximal ideals. Hence $U(\mathfrak{sl}_\infty) \not\cong U(\mathfrak{so}_\infty)$ and $U(\mathfrak{sp}_\infty) \not\cong U(\mathfrak{so}_\infty)$.

Consider now prime submaximal integrable ideals in $U(\mathfrak{sl}_\infty)$ and $U(\mathfrak{sp}_\infty)$, i.e. integrable prime ideals which are properly contained only in integrable maximal ideals. Using the inclusion criterion of irreducible c.l.s. from Subsection 7.3, one checks immediately (using Theorem 7.8 a)) that $U(\mathfrak{sl}_\infty)$ has two such submaximal ideals, namely $I(\mathcal{L}_1)$ and $I(\mathcal{R}_1)$, while $U(\mathfrak{sp}_\infty)$ has a single such ideal $I(\mathcal{L}_1)$. Hence $U(\mathfrak{sl}_\infty) \not\cong U(\mathfrak{sp}_\infty)$. \square

Finally, note that

$$\text{Var}(I(v, Q_f)) = \begin{cases} \mathfrak{sl}_\infty^{\leq v} & \text{for } \mathfrak{g}_\infty = \mathfrak{sl}_\infty, \\ \mathfrak{so}_\infty^{\leq 2v} & \text{for } \mathfrak{g}_\infty = \mathfrak{so}_\infty, \\ \mathfrak{sp}_\infty^{\leq 2v} & \text{for } \mathfrak{g}_\infty = \mathfrak{sp}_\infty. \end{cases} \quad (10)$$

Since $\mathfrak{so}_\infty^{\leq 2v+1} = \mathfrak{so}_\infty^{\leq 2v}$, all possible associated “varieties” of ideals in $U(\mathfrak{sl}_\infty)$ and $U(\mathfrak{so}_\infty)$ appear in the right-hand side of (10). The only possible associated “varieties” which do not appear in the right-hand side of (10) are $\mathfrak{sp}_\infty^{\leq 2v+1}$ for $v \in \mathbb{Z}_{\geq 0}$, and these proj-varieties are not associated “varieties” of integrable ideals of $U(\mathfrak{sp}_\infty)$.

8. SOME NON-INTEGRABLE IDEALS OF $U(\mathfrak{sp}_\infty)$

We will now provide non-integrable ideals of $U(\mathfrak{sp}_\infty)$ whose associated “varieties” are the proj-varieties $\mathfrak{sp}_\infty^{\leq 2v+1}$ for $z \in \mathbb{Z}_{\geq 0}$. We start with a lemma and a corollary.

Let \mathfrak{g} be a semisimple Lie algebra. Consider the coproduct

$$\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}) \quad (x \mapsto x \otimes 1 + 1 \otimes x \text{ for } x \in \mathfrak{g}).$$

In what follows we will denote the “diagonal”, “left” and “right” copies of \mathfrak{g} respectively by \mathfrak{g}_Δ , \mathfrak{g}_l , \mathfrak{g}_r , i.e. we have

$$\Delta : U(\mathfrak{g}_\Delta) \subset U(\mathfrak{g}_l) \otimes U(\mathfrak{g}_r).$$

Lemma 8.1. Let I_l, I_r be ideals of $U(\mathfrak{g})$. Then

$$\text{Var}(I_l) + \text{Var}(I_r) \subset \text{Var}(U(\mathfrak{g}_\Delta) / (U(\mathfrak{g}_\Delta) \cap (I_l \otimes 1 + 1 \otimes I_r))),$$

where $\text{Var}(I_l) + \text{Var}(I_r)$ is a pointwise sum of the varieties $\text{Var}(I_l)$ and $\text{Var}(I_r)$ inside \mathfrak{g}^* .

Proof. We have

$$\text{gr}(U(\mathfrak{g}_\Delta) \cap (I_l \otimes U(\mathfrak{g}_r) + U(\mathfrak{g}_l) \otimes I_r)) \subset \text{gr}(I_l \otimes U(\mathfrak{g}_r) + U(\mathfrak{g}_l) \otimes I_r) = \text{gr } I_l \otimes S^*(\mathfrak{g}_r) + S^*(\mathfrak{g}_l) \otimes \text{gr } I_r.$$

Therefore

$$\text{Var}(\text{gr}(I_l)) + \text{Var}(\text{gr}(I_r)) \subset \text{Var}(\text{gr}(U(\mathfrak{g}_\Delta) \cap (I_l \otimes U(\mathfrak{g}_r) + U(\mathfrak{g}_l) \otimes I_r))).$$

\square

Corollary 8.2. Let M_l, M_r be \mathfrak{g} -modules with annihilators I_l, I_r . Let I_Δ be the annihilator of $M_l \otimes M_r$. Then

$$\text{Var}(I_l) + \text{Var}(I_r) \subset \text{Var}(I_\Delta).$$

Proof. We first consider $M_l \otimes M_r$ as $U(\mathfrak{g}_l) \otimes U(\mathfrak{g}_r)$ -module. Then the structure of \mathfrak{g} -module on $M_l \otimes M_r$ comes from the homomorphism Δ . Therefore

$$\text{Ann}_{U(\mathfrak{g})}(M_1 \otimes M_2) = U(\mathfrak{g}_\Delta) \cap (I_l \otimes 1 + 1 \otimes I_r).$$

Hence $\text{Var}(I_l) + \text{Var}(I_r) \subset \text{Var}(I_\Delta)$ by Lemma 8.1. \square

We now provide a maximal ideal $I \subset U(\mathfrak{sp}_\infty)$ such that $\text{Var}(I) = \mathfrak{sp}(V)^{\leq 1}$.

Consider the Weyl algebra \mathbb{W}_∞ of infinitely many variables, i.e. the associative algebra generated by $x_1, x_2, \dots, x_n, \dots, \partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}, \dots$ with relations

$$\begin{aligned} [x_i, x_j] &= 0, \quad [\partial_{x_i}, \partial_{x_j}] = 0 \text{ for all } i, j; \\ [\partial_{x_i}, x_i] &= 1 \text{ for } 1 \leq i \leq n-1; \quad [\partial_{x_i}, x_j] = 0 \text{ for all } i \neq j. \end{aligned}$$

The subspace spanned by $1, x_i x_j, x_i \partial_{x_j}, \partial_{x_j} \partial_{x_i}$ is a Lie subalgebra of the Lie algebra associated with \mathbb{W}_∞ . This subalgebra $\tilde{\mathfrak{s}}$ has a 1-dimensional center generated by 1, and its derived subalgebra $[\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}]$ is isomorphic to \mathfrak{sp}_∞ . Therefore we have a homomorphism $U(\mathfrak{sp}_\infty) \rightarrow \mathbb{W}_\infty$; the kernel of this homomorphism is an ideal $I_{\mathbb{W}}$. For any n , $\mathfrak{g}_n = \mathfrak{sp}_{2n}$ and $I_{\mathbb{W}} \cap U(\mathfrak{g}_n)$ is a Joseph ideal [Jo]. We denote by $V_{\mathbb{W}}$ the algebra $\mathbb{F}[x_1, x_2, \dots]$ considered as an \mathfrak{sp}_∞ -module.

It is well known that the ideal $I_{\mathbb{W}} \cap U(\mathfrak{g}_n)$ is maximal (thus primitive and prime) in $U(\mathfrak{sp}_{2n})$ and not integrable. Hence $I_{\mathbb{W}}$ is a maximal ideal in $U(\mathfrak{sp}_\infty)$ which is not integrable. The zero-set $\text{Var}(I_{\mathbb{W}})$ coincides with $\mathfrak{sp}_\infty^{\leq 1}$ (note that any element of rank 1 in \mathfrak{sp}_{2n} is nilpotent).

More generally, we have the following.

Proposition 8.3. For any $v \in \mathbb{Z}_{\geq 0}$ we have

$$\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})) = \mathfrak{sp}_\infty^{\leq 2v+1}.$$

Proof. By Corollary 8.2, we have

$$\begin{aligned} \overline{\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v))) + \text{Var}(I_{\mathbb{W}})} &= \overline{\mathfrak{sp}_\infty^{\leq 2v} + \mathfrak{sp}_\infty^{\leq 1}} = \\ &= \mathfrak{sp}_\infty^{\leq 2v+1} \subset \text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})). \end{aligned}$$

For any n there is a natural embedding $\mathfrak{sl}(V_n) \rightarrow \mathfrak{sp}(V_n \oplus V_n^*)$, and these embeddings define an embedding $\mathfrak{sl}_\infty \rightarrow \mathfrak{sp}_\infty$. We have

$$\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}}) \cap U(\mathfrak{sl}_\infty) = I(2v+1, \mathbb{F}),$$

where \mathbb{F} stands for the c.l.s. corresponding to the augmentation ideal, and therefore the image of $\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}}))$ under the projection $\mathfrak{sp}_\infty^* \rightarrow \mathfrak{sl}_\infty^*$ lies in $\mathfrak{sl}_\infty^{\leq 2v+1}$. Hence, by Lemma 4.9,

$$\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})) \subset \mathfrak{sp}_\infty^{\leq 2v+1}.$$

This shows that $\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}^\cdot(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})) = \mathfrak{sp}_\infty^{\leq 2v+1}$. \square

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